

Are moving punctures equivalent to moving black holes?

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When simulating the inspiral and coalescence of a binary black-hole system, special care needs to be taken in handling the singularities. Two main techniques are used in numerical-relativity simulations: A first and more traditional one “excises” a spatial neighborhood of the singularity from the numerical grid on each spacelike hypersurface. A second and more recent one, instead, begins with a “puncture” solution and then evolves the full 3-metric, including the singular point. While the first approach is mathematically and numerically well-defined, the second one still maintains a non-differentiable point within the black hole. No strong-field evidence has yet been provided to show that the two approaches are indeed dynamically equivalent. To address this question we have used both techniques to evolve a binary system of equal-mass non-spinning black holes and compared the evolution of two curvature 4-scalars with proper time along the invariantly-defined worldline midway between the two black holes. Using Richardson-extrapolation techniques to reduce the influence of the finite-difference truncation error, we find that the moving-punctures and excision evolutions produce the same spacetimes along that worldline. This represents the first strong-field and dynamical evidence that the moving-puncture prescription is robust both mathematically and numerically.

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Introduction. Binary black hole coalescences are both natural laboratories in which to study the nonlinear strong-field dynamics of General Relativity and among the most promising sources of gravitational radiation for modern laser-interferometric detectors. Despite these being very simple systems, as the black holes are assumed to be in vacuum and the solution of the Einstein equations fully describes the binary, no analytic solutions are known and numerical methods represent the only viable approach to investigate the dynamics of the system. The past few years have seen major advances in these numerical simulations, with demonstrations of multiple orbit evolutions through merger [1, 2, 3, 4, 5], recoils from unequal-mass systems [6, 7], and studies of spin couplings in the final orbit [8, 9, 10]. Convergence studies and cross-checks between independent codes [11] have demonstrated an impressive consistency, lending support to their credibility as reliable modellers of these important sources.

Much of the recent success results from some significant changes in the methodology of numerical black hole simulation. Whereas previous codes attempted to excise the singular point within a black hole (a chronic source of trouble for numerical simulations), it was recently demonstrated that this technically involved approach can be dispensed with if the black hole initial data is determined by a construction commonly called “puncture data” [12] and evolved *without excision* using suitable gauges [13], potentially even allowing the singularities to be advected across the computational grid [14, 15]. Following a different line of reasoning, [16] have recently shown that the combination of not using excision, suitable gauge conditions, and minute numerical dissipation can dramatically improve the long-term stability of simulations of gravitational collapse to rotating black holes, allowing for the calculation of complete waveforms.

Many new results are now being produced from the moving-puncture method, so it is worth examining it in some detail. We recall that by this method the curvature singularity at the centre of a black hole is avoided and replaced by an asymptotically flat spacetime through the throat. A coordinate singularity at the effective $r = 0$ of each black hole still remains, and this represents a non-differentiable point which, at least in principle, needs special treatment. Standard finite-difference techniques, in fact, require smooth functions at each gridpoint and thus would not be able to evaluate derivatives in the neighbourhood of the puncture. In practice, however, the inaccuracies at these points are isolated and the physical causality of the spacetime ensures that errors at the singularity do not propagate into the observable part of the spacetime. Finite-differencing respects this causality provided a sufficient number of computational points exists between the black hole horizon and the singularity, and common experience shows that if a sensible representation of the Einstein equations is used to carry out the evolution, the inaccuracies near the singularities do not lead to numerical instabilities.

In addition to a suitable causal structure, an important property of punctures that seems to be relevant in establishing their validity in practice is that standard singularity-avoiding gauge conditions lead to spacetimes that are essentially stationary in their neighbourhood. This has been pointed out in [16] and more extensively discussed in [17], where it has been shown that coordinate conditions already commonly used in numerical relativity lead to a stationary slicing of the Schwarzschild spacetime where the moving-puncture coordinate singularity at $r = 0$ corresponds to a sphere of finite areal radius (located *inside* the apparent horizon).

These fortunate properties of black hole spacetimes in general, and of punctures in particular, can remove the need for

complicated excision techniques, which are prone to numerical problems when a topologically spherical surface is excised in Cartesian coordinates. However, the failure to satisfy the Einstein equations in the strong-field neighbourhood of the singularity raises some important questions. Do these solutions represent physically realistic spacetimes? Can these regions be accurately advected? Stated differently: Are moving punctures equivalent to moving black holes?

One way of answering these questions is to compare puncture methods with alternate means of evolving black-hole solutions, in particular with those employing excision techniques. These, we recall, replace the core of the black hole by an inner boundary condition at a finite radius [18, 19] and appeal to the same causality properties mentioned above for punctures: if the black hole exterior is protected by a horizon, it will be independent of any manipulation of the interior. Indeed, the idea of replacing the troublesome black hole interior with an excised region has been a central paradigm of numerical relativity for over a decade. The technical implementation of a stable excision algorithm in Cartesian coordinates has proven a challenge for binary systems and has only recently been overcome. In particular, binary punctures were first evolved using excision and yielded the first evolution for a timescale of an orbit [20], with accurate trajectories determined in [21]. In both cases, the implementation of the excision technique was significantly simplified by the use of corotating coordinates, so that the excision boundary remained fixed on the computational grid [22]. More recently, however, implementations of moving excision-domains have also been presented [23] and used in binary simulations [1, 24].

Comparing evolutions of binary systems using either corotating excision or moving punctures raises a number of challenges, most notably, determining which quantities can be compared. Clearly, these must be gauge-invariant so as not to depend on the details of the two different calculations but, if asymptotic, also be physically significant such as the gravitational-wave signals. Work carried out in [25] has indeed demonstrated that in the case of head-on collisions pure puncture evolutions produce essentially identical waveforms as their excised counterparts. In the case of an orbiting system, however, such an accurate comparison is not yet possible in practice. While a number of accurate waveforms have been presented using moving-punctures, the evolutions of corotating and excised punctures have not produced usable asymptotic waveforms [20, 21] due to technical complications in wave extraction when using corotating coordinates ([26] presents a possible route to overcoming these problems.) As a result, any comparison must rely on quantities measured invariantly in the strong-field and highly-dynamical regions of spacetime.

The object of this paper is to perform such a comparison and demonstrate that evolutions of non-spinning puncture data yield identical results when using the moving-puncture method or when excising the punctures in a corotating gauge. We do this by comparing the evolution of two curvature invariants measured along geodesics and by noting that in the case of equal-mass binaries, the geodesic located half-way between the two black holes is defined invariantly. Although

we concentrate on the evolution of a single point in spacetime, i.e., the only one allowing for such a comparison, the results presented here provide the first strong-field evidence that moving-puncture can be used reliably to describe the dynamics of binary black-hole systems.

Methods and Results. All the numerical simulations for both corotating excision (hereafter “CE”) and moving punctures (hereafter “MP”) have been performed using the same evolution code and initial data. The latter, in particular, are constructed as in [27] and have orbital parameters to approximate a binary system of non-spinning black holes in quasi-circular orbit, with initial separation $L = 9.32M$, mass parameters $m = 0.47656M$, where M is the total mass of the system, and equal and opposite linear momenta $p = \pm 0.13808M$ [20]. The evolutions are carried out using a conformal-traceless formulation of the Einstein equations as described in [13], with “1+log” slicing and Γ -driver shift. The CE runs benefit from insights gained in [21] and use the GC2 gauge condition of that work. The MP runs use the optimal gauge conditions of [28], with the lapse evolved via $\partial_t \alpha = -2\alpha K + \beta^i \partial_i \alpha$, while the shift evolution follows prescription 8 in Table I of [28] with $\eta = 0.5$. Individual apparent horizons are located every few timesteps during the evolution [29, 30].

Spatial differentiation is performed via straightforward finite-differencing using second- or fourth-order algorithms for CE and MP, respectively. In addition, for the MP runs a fifth-order Kreiss-Oliger artificial dissipation is also added to all evolution variables. Vertex-centered AMR is employed using nested mesh-refined grids [31] with the highest resolution concentrated in the neighbourhood of the individual horizons. In the case of CE evolutions, eight levels of refinement have been used; the corotating gauge conditions guarantee that the black holes remain on the fine grids throughout the evolution. In the case of MP evolutions, on the other hand, nine levels of refinement are used, with the finest two levels being locked to the position of the centroid of the apparent horizon. For either the CE or MP approach, we have carried out simulations with at least three different resolutions. However, because the two approaches have rather different truncation errors, with MP being intrinsically more accurate, the CE simulations have been carried out with fine-grid spatial resolutions of $h = 0.018, 0.015$, and $0.0125M$, while the MP ones have generically coarser resolutions, with $h = 0.032, 0.025$, and $0.020M$.

As mentioned earlier, an unambiguous measure of the CE and MP spacetimes can be made by using the 4-invariant spacetime curvature scalars $I \equiv \tilde{C}_{\alpha\beta\gamma\delta} \tilde{C}^{\alpha\beta\gamma\delta}$ and $J \equiv \tilde{C}_{\alpha\beta\gamma\delta} \tilde{C}^{\gamma\delta}_{\mu\nu} \tilde{C}^{\mu\nu\alpha\beta}$, where $\tilde{C}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + \frac{1}{2} i \epsilon_{\alpha\beta\mu\nu} C^{\mu\nu}_{\gamma\delta}$ is the self-dual part of the Weyl tensor $C_{\alpha\beta\gamma\delta}$. Note that while I and J are complex numbers, for our evolutions their real parts are at least 12 orders of magnitude larger than the imaginary ones, so that $I, J = \Re(I, J)$ to very good precision. Hereafter we will concentrate on reporting results for I only, as a very similar behaviour was found also for J .

The measure of the invariants has to be made along exactly the same worldline in the two spacetimes, and when the black holes have equal masses, the spatial point midway between the black holes is invariantly defined, and its worldline can

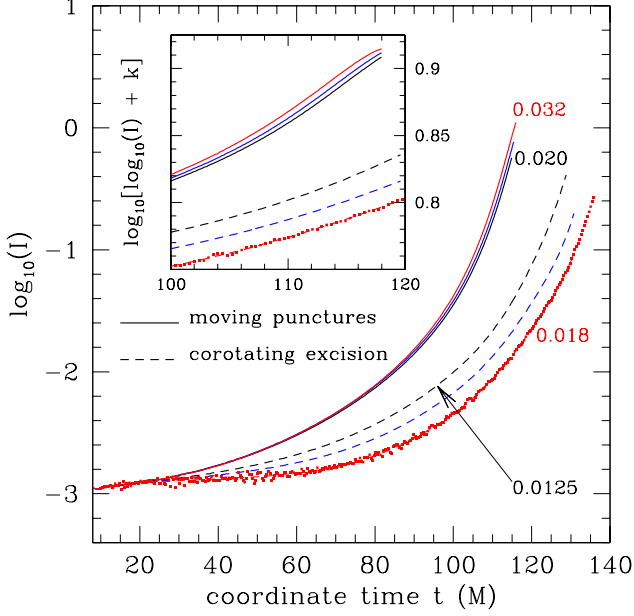


FIG. 1: Evolution in coordinate time of $\log I$ for the MP evolutions (solid lines) and the smoothed CE ones (dashed lines), while the raw CE-data is indicated with squares for the coarsest resolution only. A magnification of the overlapping MP curves is shown in the inset.

be used for this measure. Clearly, evolutions with different gauges will generate different coordinate descriptions of this point, but this ambiguity is absent when the affine parameter along the geodesic is chosen to be the proper time τ . As a result, I expressed as a function of τ along the worldline of the midway point between the two black holes can be used as a gauge-invariant diagnostic of the evolution. We note that because $I(\tau)$ has a super-exponential behaviour, all of the analysis has been performed in terms of $\log I$ to increase accuracy.

Figure 1 shows the evolution in coordinate time of $\log I$ for the MP evolutions (solid lines) and the smoothed CE ones (dashed lines). The raw I -timeseries for the CE evolutions are quite noisy (cf. small squares in Fig. 1), and for further analysis we smoothed these with a fourth-order Savitzky-Golay filter [32] over a $\pm 10 M$ sliding window in coordinate time. We have verified the smoothing does not introduce systematic errors; no smoothing was necessary for the MP evolutions.

Even with modern supercomputers, both MP and CE evolutions remain computationally demanding and it is not practical to make h small enough so that finite-differencing errors are negligible. Instead, we exploit the known convergence properties of finite-difference schemes to Richardson-extrapolate our finite- h results to the limit $h \rightarrow 0$. In particular, given some quantity u computed at numerical resolution h , we write the Richardson-extrapolation series $u(h)$ as

$$u(h) = u(0) + ph^n + qh^{n+1} + \mathcal{O}(h^{n+2}), \quad (1)$$

where $n=2(4)$ for CE (MP), and where the coefficients p and q depend on u , but not on the resolution h . Given $u(h)$ at three distinct resolutions, we solve for $u(0)$ as the Richardson-extrapolated value for u , i.e., $\mathcal{R}(u) \equiv u(0)$. Clearly, slightly

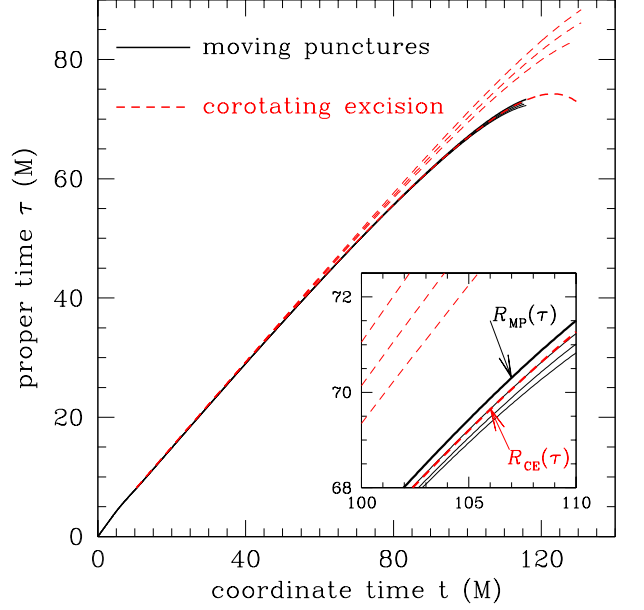


FIG. 2: Relationship between coordinate time t and proper time τ for simulations using MP (solid lines) or CE (dashed lines). The inset offers a magnification over a representative window in time.

different values for $\mathcal{R}(u)$ will be obtained depending on which of the higher-order terms are neglected in the series expansion (1), and we use the magnitude of the last known term in (1) at our highest resolution as a rough estimate of the numerical errors in $\mathcal{R}(u)$.

In practice, for each evolution we have first extracted the timeseries of α and I up to the detection of a common apparent horizon and then time-integrated $\alpha(t)$ to obtain $\tau(t)$, as shown in Fig. 2 for simulations using MP (thin solid lines) or CE (thin dashed lines). Using this data and the series expansion (1), a Richardson-extrapolated estimate for $\mathcal{R}(\tau(t))$ is then obtained and shown with thick lines (solid for MP and dashed for CE), with the inset offering a view. Note that despite having lower resolutions, the MP evolutions show a much closer match between the different resolutions and the Richardson-extrapolated result than do the CE evolutions.

Finally, we have Richardson-extrapolated $\log I(t)$, and removed the dependence on the time coordinate by mapping t to $\mathcal{R}(\tau(t))$ (cf. Fig. 2). Our end results are therefore $\mathcal{R}(\log I_{\text{CE}})$ and $\mathcal{R}(\log I_{\text{MP}})$, both as functions of $\mathcal{R}(\tau)$.

The results of this procedure are summarized in Fig. 3, which shows the proper-time evolution of $\log I(\tau)$, together with the estimated error bands. More specifically, thick lines show the Richardson-extrapolated results (solid for MP and dashed for CE) while the dotted lines report the error bars, with the larger ones referring to CE evolutions. Clearly, the two Richardson-extrapolated evolutions of the invariant lie well within the estimated error-bands for both evolutions and are almost indistinguishable for large portions of the simulations, despite the large dynamical range. The inset highlights this, with a view in a representative window in proper time.

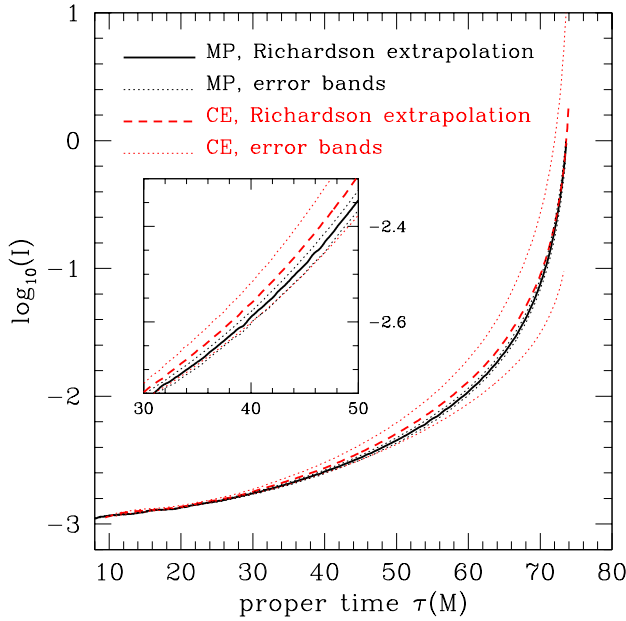


FIG. 3: $\log I(\tau)$ for each evolution family, together with the estimated errors. Thick lines show the Richardson-extrapolated results (solid for MP and dashed for CE) while the dotted lines report the error bars, with the larger ones referring to CE evolutions. Note the excellent agreement as highlighted in the inset.

Overall, the results in Fig. 3, together with the similar ones for J , demonstrate that, despite the different gauges and the different way in which the singularities are treated in the two approaches, the two approaches are indeed converging to the same spacetime, at least along the fiducial central geodesic.

Conclusions. Moving punctures have rapidly become a standard approach to simulate the dynamics of binary black holes. However no strong-field evidence has yet been presented that moving punctures are indeed dynamically equivalent to moving black holes. By evolving an equal-mass binary black-hole system both as moving punctures and with the more traditional excision technique, we have here demonstrated that this is indeed the case. More specifically, we have shown that the Richardson-extrapolated evolution of the I and J curvature 4-scalars along the invariantly-defined worldline between the two black holes is identical in the two cases, up to the estimated numerical errors. Although specific to non-spinning black holes, these results offer the first evidence, from a curved and highly dynamical region of spacetime, that the moving-puncture prescription is indeed equivalent to that of excised moving black holes.

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